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# Oscillations *vs.* chaotic waves: Attractor selection in bistable stochastic reaction–diffusion systems

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**Abstract.** In bistable systems, the long-term behavior of solutions depends on the location of the initial conditions. In a deterministic setting, where the initial condition is kept fixed in one particular basin of attraction, repeated numerical simulations will always lead to the same long-term behavior. The other possible asymptotic solution type will never be observed. This clear distinction does not hold anymore if the system is forced by random fluctuations. In this case, both asymptotic solutions can be attained, and the relative frequency of different long-term behaviors observed in many repeated simulation runs will follow a certain probability distribution. We present a simple reaction–diffusion model of spatial predator–prey interaction, where depending on the initial spatial distribution of the two populations either spatially homogeneous or spatiotemporal irregular oscillations may be observed. We show by repeated stochastic simulations that, when starting in the basin of attraction of the spatiotemporal irregular solution, in the randomly forced system the probability to observe spatially homogeneous oscillations instead of spatiotemporally irregular oscillations follows a non-trivial bimodal distribution.

## 1 Introduction

Noise is an important component in the modelling of population dynamics, since natural populations are inevitably subject to internal demographic and external environmental fluctuations. Among the more obvious effects of such random fluctuations in models of population dynamics is that noise blurs unrealistic distinct and symmetric spatial and spatiotemporal patterns. In some cases, however, noise may induce transitions, sometimes catastrophic shifts, between different dynamical regimes of the model. Examples of such shifts include noise-triggered and -influenced transitions in bi- and multistable systems [1–6], sustained oscillations in otherwise stationary systems, which may give rise to regular spatiotemporal patterns [7–10] and the suppression of periodic travelling waves in oscillatory reaction–diffusion systems [11]. All of these mentioned studies deal with noise-induced shifts that occur well after the respective system has settled on a particular long-term behavior.

However, in this paper we want to briefly discuss an effect of noise which is exclusively associated with the transient phase of solutions to a reaction–diffusion model of predator–prey interaction. Here by transient phase we mean the period of time, during which the solution, starting from a particular initial condition, approaches an attractor of the system before eventually settling on that attractor. Depending on the initial condition, this transient phase can have considerable length and thus noise may have profound impacts.

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## 2 Model and simulations

### 2.1 The model

In the following, let $u = u(t, x)$ and $v = v(t, x)$ denote the densities of the prey and predator species at time $t$ and spatial location $x \in [0, L]$ on a spatially one-dimensional domain of length $L$. Dispersal of both species is assumed to be adequately described by diffusion and the spatially homogeneous diffusion coefficient $D$ is the same for both species. In the following, we will restrict our attention to Neumann no-flux boundary conditions where $\partial u/\partial x = \partial v/\partial x = 0$ at $x = 0$ and $x = L$. The domain length will be fixed to $L = 1200$ together with a diffusion coefficient of $D = 1$. A specific model for the interaction and dispersal of two species subject to noise is given by the two coupled stochastic reaction–diffusion equations

$$\frac{\partial u}{\partial t} = D\,\frac{\partial^2 u}{\partial x^2} + \left( r\,(1-u) - \frac{a\,v}{h+u} \right)\, u + \eta\,\xi_t\,u, \tag{1}$$

$$\frac{\partial v}{\partial t} = D\,\frac{\partial^2 v}{\partial x^2} + \left( \epsilon\,\frac{a\,u}{h+u} - m \right)\, v + \eta\,\xi_t\,v. \tag{2}$$

The first terms on the right-hand side of the equations form the deterministic skeleton of the model, here given by the well-known predator–prey model due to Rosenzweig and MacArthur [12]. The prey population $u$ grows logistically with intrinsic growth rate $r$ and a carrying capacity, which is scaled to unity in the above formulation. Predation is modeled via a Holling-type II functional response [13] with predation rate $a$ and half saturation constant $h$, which is characteristic for specialist predators preying on a single prey species. Prey consumption is scaled by the efficiency parameter $\epsilon$ to yield the numerical response of the predator, while predators die with per-capita mortality rate $m$. This Rosenzweig–MacArthur model is well-known to exhibit sustained oscillations following a Hopf-bifurcation of the unique strictly positive stationary solution $(u^*,\, v^*)$. A typical parameter set yielding oscillatory behavior which will be used from now on is

$$r = 1, \quad a = 1, \quad h = 0.3, \quad \epsilon = 2, \quad m = 0.8.$$

Noise is introduced into the model via the real-valued stochastic process $\xi_t$, which is the same for both species since they occupy the same habitat and thus experience the same noisy fluctuations. This stochastic process can be viewed as representing all environmental processes for which an impact on the system dynamics can not be ruled out, but which are not explicitly modeled by the deterministic skeleton of the model. This could be the case for processes that are too poorly understood to allow for explicit description and/or for processes that are overly complex [14]. Examples of such processes are rapidly changing atmospheric conditions such as temperature, humidity, precipitation or wind stress; other species that interact with the modeled species; the availibility of basic resources; the movement and salinity of the surrounding water column for aquatic species; intensity of light irradation for photosynthetic species. The constant parameters in the deterministic skeleton are usually assumed to reflect the long-term mean values of these processes. However, to account for the inevitable fluctuations around this mean, we will assume that the noise process $\xi_t$ has a joint normal distribution with zero mean and unit variance, which does not change with time. Further, we require that the elements of this stationary noise process $\xi_t$ at different times are uncorrelated. A stochastic process fulfilling these requirements is commonly known as Gaussian white noise. Such white noise has been widely used as an approximation to such processes, if they can be assumed to fluctuate rapidly on time scales that are small compared to the time scales of interaction and dispersal of the modeled species [15]. The analysis of data sets of environmental variables indicates that this assumption is acceptable for many terrestrial ecosystems [16,17]. Note that the stochastic term on the right hand side of Eqs. (1)–(2) is essentially multiplicative white noise, which is zero if there is no population at all. This reflects the postulate of parenthood formulated by Hutchinson [18]. Note also that we assume that the habitat is small enough so that spatial variations of the fluctuations can

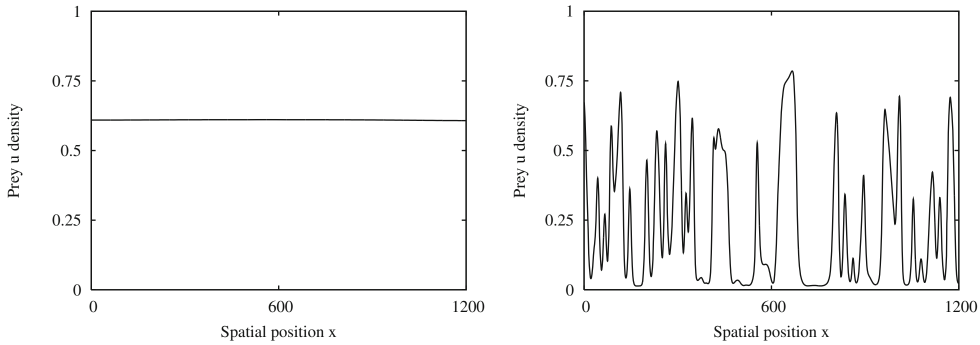


**Fig. 1.** Starting from initial conditions (3), with stochasting forcing the two qualitatively different long term behaviors are approached with probability $\phi$ (homogeneous solution, left) and $1 - \phi$ (irregular solution, right), respectively.

be neglected. Thus, $\xi_t$ does not depend on space, but rather is a purely temporal stochastic process. The magnitude of the stochastic forcing is scaled by the common noise strength $\eta$, which will be the control parameter in the following.

## 2.2 Stochastic simulations

It has been shown by Petrovskii and Malchow [19] that depending on the initial spatial distribution of $u$ and $v$ the deterministic system (1)–(2) may approach two qualitatively different long term solution types. Either, spatially homogeneous oscillations emerge after a sometimes very long transient phase, or spatiotemporal irregular oscillations spread over the whole spatial domain which then persist indefinitely. This latter scenario has been termed "wave of chaos" by Petrovskii and Malchow [19], due to the essentially chaotic dynamics at fixed spatial locations. The two possible scenarios are shown in Fig. 1, the spatially homogeneous oscillations on the left and the persistent spatiotemporal irregular oscillations on the right. This phenomenon has also been investigated by Sherratt [20] in the more generic setting of oscillatory reaction–diffusion systems of $\lambda$–$\omega$-type. These irregular oscillations persist indefinitely and are not suppressed by low to moderate spatiotemporal fluctuations [11].

In summary, in a deterministic setting with $\eta = 0$ the position of the initial condition with respect to the basins of attraction of the two possible attracting solution types completely determines the long term behavior of the corresponding particular solution to Eqs. (1)–(2). For noise intensities $\eta > 0$ however, the long term behavior of a solution can not be predicted with certainty, especially if the initial condition is close to the boundary of the two basins of attraction. In the following we will denote the basin of attraction of the spatially homogeneous oscillations as the *homogeneous basin* and the one of spatiotemporal irregular oscillations as the *irregular basin*.

Assume now that an initial condition is given that leads to spatiotemporal irregular behavior in the deterministic case $\eta = 0$, corresponding to an initial condition in the irregular basin. Let now $\phi$ denote the probability that spatially homogeneous oscillations arise instead of spatiotemporal irregular oscillations. This probability will in general depend on the strength of the stochastic forcing, that is $\phi = \phi(\eta)$, and we can expect $\phi(\eta) > 0$ for some $\eta > 0$ if the initial condition is close to the boundary with the homogeneous basin.

As an example, we consider the initial condition

$$\begin{aligned} u(0,\, x) &= u^*, \\ v(0,\, x) &= v^* + \alpha\,(x - x_0), \end{aligned} \tag{3}$$

with $x_0 = L/2$ and $\alpha = 10^{-4}$, which is similar to the initial condition used in Petrovskii and Malchow [19]. This corresponds to a very slight linear spatial gradient in the initial predator

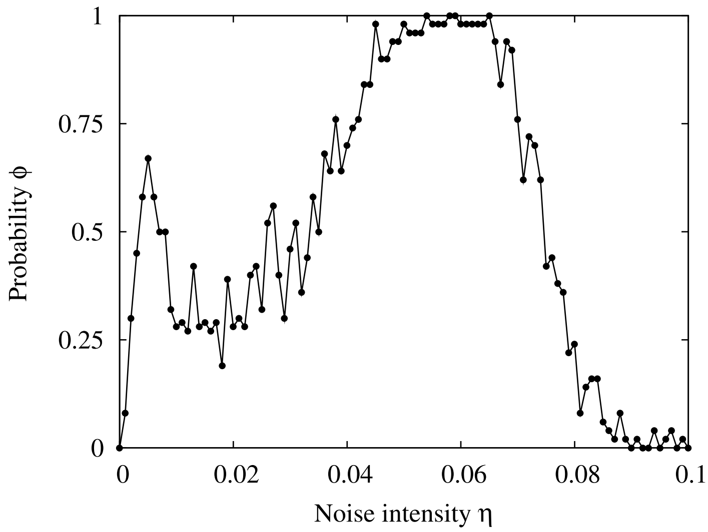


**Fig. 2.** Probability $\phi$ vs. noise intensity.

distribution, while the prey density is simply set to its equilibrium value across the entire domain. Note that the system's equilibrium value is attained at $x_0$. In a deterministic setting with $\eta = 0$, the solution starting at this particular initial condition eventually evolves into irregular spatiotemporal oscillations, as shown on the right in Fig. 1. Thus, we have an initial condition for which $\phi(0) = 0$ holds.

Now we want to study how $\phi(\eta)$ behaves for rising noise intensity $\eta$. Therefore, we choose 100 noise intensities equally distributed over the interval [0, 0.1] and then numerically integrate the system using the Euler-Maruyama method [21] for the stochastic reaction term combined with a Crank-Nicholson scheme for the diffusion term [22]. For every noise intensity $N$ independent simulations up to $t = t_{max}$ are performed and the numerical solutions are then classified according to the particular long-term behavior. Thus we obtain $n_\eta$ spatially homogeneous oscillations and $m_\eta$ spatiotemporally irregular oscillations with $n_\eta + m_\eta = N$ for each noise intensity and thus we can estimate $\phi(\eta)$ by

$$\phi(\eta) \approx \frac{n_\eta}{N}.$$

We choose $t_{max} = 25.000$, which is large enough even for the longest transient phases, and $N = 100$, which already gives a good approximation of the behavior of $\phi(\eta)$.

The results of the stochastic simulations are shown in Fig. 2. Clearly, the probability $\phi$ that spatially homogeneous oscillations arise shows two distinct maxima, one at low noise intensity and one at intermediate noise intensity. The two maxima are separated by a minimum where the spatiotemporal irregular solution is approached more often.

## 3 Discussion

Intuitively, one might expect a unimodal behavior of $\phi(\eta)$ with respect to noise intensity $\eta$, with $\phi$ strictly increasing at low noise intensity reflecting the increasing probability of an early shifting of the initial condition from the irregular basin to the homogeneous basin. After this initial increase the growing influence of the noisy forcing and its tendency to amplify spatial inhomogeneities can be expected to lead to a subsequent decline in $\phi$, thus leading to a single maximum of $\phi$ at intermediate noise intensities.

The actual situation however appears to be more complicated, since the stochastic simulations indicate that there are two intervals of noise intensities which are optimal in terms of chaos prevention. It is not yet clear why $\phi$ shows the mentioned bimodal behavior, but repeated simulations with initial distributions other than those considered here and also incorporating stochastic terms other than multiplicative noise show at least a bimodal shape [23]. So this effect seems to be a quite robust feature of the stochastic model that has been considered here.

It is an interesting question for future work, how the observed probability distribution changes qualitatively with respect to changes in properties of the underlying stochastic reaction–diffusion system. Especially interesting is the question, how the form of the initial conditions relates to the probability to observe a particular long-term behavior in the stochastic system.

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